

# Quantum Computing without Entanglement

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## Abstract

We show that a single qutrit is enough to implement a quantum algorithm which can solve a model problem faster than any classical algorithm. For permutation functions defined on a set of three elements, deciding whether a given permutation is even or odd, requires evaluation of the function for two elements. We demonstrate that a quantum circuit with a single qutrit can determine the parity of the permutation with only one evaluation of the function instead of two. Since a qutrit is the simplest system where contextual nature of quantum mechanics is observed, we conclude that contextuality might play an important role in quantum computation. The algorithm can be generalized to higher dimensional qudits with the same speed-up ratio.

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## I. INTRODUCTION

Deutsch's algorithm is not only the first quantum algorithm but also it is the simplest one [1]. Even though in its original form, algorithm of Deutsch worked probabilistically, its improvement to a deterministic algorithm is not difficult [2]. The Deutsch algorithm involves two qubits and it distinguishes constant functions, which take both input values (0 or 1) to a single output value, from the balanced functions in which output values are different. In this work we introduce an even simpler algorithm which makes use of only a single qutrit to determine the parity of permutations of a set of three objects. As in the case of Deutsch's algorithm, we obtain a speed-up relative to corresponding classical algorithms.

What makes quantum algorithms interesting is that they can solve some problems faster than classical algorithms. Deutsch coined the term quantum parallelism to stress the ability of a quantum computer to perform two calculations simultaneously. How simple can a quantum circuit be? Or, what is the smallest quantum processor that can solve a problem faster than any classical algorithm? A closely related question is the origin of the power of quantum computation. Superposition, entanglement and discord are known to play essential roles in quantum computing and yet the origin of the power of the quantum algorithms is not completely clear [3].

Kochen and Specker [4] and also Bell [5] proved that quantum mechanics can be completed only by a contextual hidden variable model. Quantum systems are contextual since a particular outcome of a measurement cannot be understood as revealing the pre-existing definite value of some underlying hidden variable [6]. Like superposition, entanglement and discord, we might ask if contextuality plays any role in quantum computation. Recently, it has been argued that quantum contextuality is a critical resource for quantum speed-up of a fault-tolerant quantum computation model [7]. Here, we present an explicit example where a contextual but unentangled system can be used to solve a problem faster than classical methods.

Superposition of states is a phenomenon that can be observed in any quantum system starting from qubits however there is no known algorithm for a single qubit. Smallest system where we can talk about entanglement is a two-qubits system which can be used to run Deutsch's algorithm. For a single qutrit, we cannot talk about entanglement. Recently, Klyachko, Can, Binicioğlu and Shumovsky have developed the simplest test to demonstrate

the contextuality of qutrit system [8]. Finding a quantum algorithm for a single qutrit provides evidence for the role of contextuality in quantum computation.

In this work, we present an algorithm that solves a black-box problem. The black-box maps three possible inputs to three possible outputs after a permutation. The six possible permutation functions of three objects are divided into two groups according to whether permutation involves odd or even number of exchange operations. Later, we are going to see that our algorithm is based on cyclic permutations but for three objects all permutations are cyclic. The computational task is to determine the parity (oddness or evenness) of a given permutation. A classical algorithm requires two queries to the black-box. We show that a quantum algorithm can solve the problem with single query. Even though the problem that the algorithm solves is not an important one, the algorithm is interesting in that it makes use of a single qutrit which means that there is no entanglement.

## II. COMPUTATIONAL TASK AND THE QUANTUM ALGORITHM

Let us consider the six permutations of the set  $\{1, 2, 3\}$ , namely  $(1,2,3)$ ,  $(2,3,1)$ ,  $(3,1,2)$ ,  $(3,2,1)$ ,  $(2,1,3)$ , and  $(1,3,2)$ . According to the parity of the transpositions, first three are even while last three are odd permutations. Our computational task is to determine the parity of a given permutation. If we treat permutation as a function  $f(x)$  defined on the set  $x \in \{1, 2, 3\}$ , determination of parity requires evaluation of  $f(x)$  for two different values of  $x$ . We are going to show that there exists a quantum algorithm where only one function evaluation is enough (instead of two) to identify whether  $f(x)$  describes an even permutation or odd.

Since we are going to use standard spin operators in our discussion let us denote the three states of a qutrit by  $|m\rangle$  where  $m = 1, 0, -1$  are the eigenvalue of  $S_z$  with  $S_z|m\rangle = m|m\rangle$ . Instead of the permutations of the set  $\{1, 2, 3\}$ , we can then consider permutations of possible  $m$  values. Our aim is to determine the parity of the bijection  $f : \{1, 0, -1\} \rightarrow \{1, 0, -1\}$ . We define the three possible even functions  $f_k$  using Cauchy's two-line notation as

$$f_1 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}, f_3 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad (1)$$

and the three odd ones as

$$f_4 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, f_6 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}. \quad (2)$$

Being a simple transposition of orthonormal states  $|m\rangle$ , operator  $U_{f_k}$  corresponding to function  $f_k$  is unitary and can easily be implemented. It is clear that direct application of  $U_{f_k}$  on basis states does not bring any improvement on classical solution. We still need to know the result of  $U_{f_k}|m\rangle$  for two different values of  $m$ . However, quantum gates can act on superposition states including

$$|\psi_1\rangle = \frac{\exp(i2\pi/3)|1\rangle + |0\rangle + \exp(-i2\pi/3)|-1\rangle}{\sqrt{3}}. \quad (3)$$

The state vector  $|\psi_1\rangle$  can be obtained from  $|1\rangle$  by single qutrit Fourier transformation, which is a generalization of Hadamard gate, defined by

$$U_{FT} = \frac{1}{\sqrt{3}} \begin{pmatrix} \exp(i2\pi/3) & 1 & \exp(-i2\pi/3) \\ 1 & 1 & 1 \\ \exp(-i2\pi/3) & 1 & \exp(i2\pi/3) \end{pmatrix}, \quad (4)$$

in  $S_z$ -basis and as we are going to show, it can be used to distinguish even and odd  $f_k$ 's. We first note that the state vectors defined by

$$|\psi_k\rangle \equiv U_{f_k}|\psi_1\rangle = \frac{\exp(i2\pi/3)|f_k(1)\rangle + |f_k(0)\rangle + \exp(-i2\pi/3)|f_k(-1)\rangle}{\sqrt{3}}, \quad (5)$$

have the property that  $|\psi_1\rangle = \exp(-i2\pi/3)|\psi_2\rangle = \exp(i2\pi/3)|\psi_3\rangle$  and  $|\psi_4\rangle = \exp(-i2\pi/3)|\psi_5\rangle = \exp(i2\pi/3)|\psi_6\rangle$ . In other words, application of  $U_{f_k}$  on  $|\psi_1\rangle = U_{FT}|1\rangle$  gives  $|\psi_1\rangle$  for even  $f_k$  and it gives  $|\psi_4\rangle = U_{FT}|-1\rangle$  for odd  $f_k$ . Therefore, if we apply the inverse Fourier transformation  $U_{FT}^\dagger$  on  $|\psi_k\rangle$ , depending on the parity of the permutation function we end up with state  $|1\rangle$  (even  $f_k$ ) or  $|-1\rangle$  (odd  $f_k$ ). Thus, single evaluation of the function is enough to determine its parity.

In summary, quantum circuit of the algorithm involves just three gates visited by a single qutrit. We start with  $|1\rangle$  and place  $U_{FT}$ ,  $U_{f_k}$ , and  $U_{FT}^\dagger$  next to each other. Final state of the qutrit after  $U_{FT}^\dagger$  gate is necessarily either  $|1\rangle$  or  $|-1\rangle$ , while  $|0\rangle$  is never observed.

### III. CONTEXTUALITY OF QUTRIT

Although we can modify our algorithm for a single qubit, where Fourier transformation becomes an Hadamard operator, it is not interesting since in this case classical solution requires single evaluation of the permutation function so that quantum algorithm does not provide any speed-up. The above algorithm is probably the simplest quantum algorithm. We might then ask what makes a qutrit special or we might wonder the origin of the computational power of qutrit.

A qutrit can be interpreted as half-way between one and two qubits. This interpretation can be made more concrete as follows. For a spin-1 system, squares of components of spin operator commute. For example, let us consider  $S_x^2$  and  $S_y^2$  where  $[S_x^2, S_y^2] = 0$ . Simultaneous eigenstates  $|pq\rangle$  (with  $p, q \in \{0, 1\}$ ) of  $S_x^2$  and  $S_y^2$  resemble the computational basis of two qubits. However, due to the constraint  $S_x^2 + S_y^2 + S_z^2 = 2$ , allowed states are  $|01\rangle$ ,  $|10\rangle$ , and  $|11\rangle$  with  $|00\rangle$  missing.

Spin operators provide an alternative interpretation of the algorithm. After the application of  $U_{f_k}$ , qutrit is known to be either in state  $|\psi_1\rangle$  or in state  $|\psi_4\rangle$ . Let us consider the Hermitian operator  $M = |\psi_4\rangle\langle\psi_4| - |\psi_1\rangle\langle\psi_1|$ . Possible states  $|\psi_1\rangle$  and  $|\psi_4\rangle$  are eigenstates of  $M$  with eigenvalues -1 and 1, respectively. We can write  $M$ , in terms of spin operators, as

$$M = \frac{i}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \sqrt{\frac{2}{3}} S_y - \frac{1}{\sqrt{3}} (S_x S_y + S_y S_x). \quad (6)$$

Therefore, if the qutrit enters a medium, just after  $U_{f_k}$  gate, so that the Hamiltonian operator is  $M$ , energy measurement will give the information about the parity of  $f_k$  directly.

Beyond being between one and two qubits, a qutrit is the smallest system where contextual nature of quantum mechanics is observed. The standard method to recognize contextual correlations is through the violation of a noncontextuality inequality. Klyachko, Can, Binicioğlu and Shumovsky introduced a 5-ray inequality which is satisfied by all non-contextual hidden variable models. KCBS inequality can be violated when state dependent five measurement directions are properly chosen. Using thirteen projectors, Oh and You demonstrated the contextual behavior of three level systems in a state independent way [9]. Therefore, all qutrit states are contextual. In other words, computational power of qutrit

can be related to contextuality.

#### IV. POSSIBLE GENERALIZATIONS

We can generalize the algorithm to  $d$ -dimensional (or equivalently spin- $(d-1)/2$ ) systems. In this case, the algorithm can be used to distinguish cyclic permutations according to their parity. For example, when  $d = 4$  positive cyclic permutations of  $(1, 2, 3, 4)$  are  $(2, 3, 4, 1)$ ,  $(3, 4, 1, 2)$  and  $(4, 1, 2, 3)$  while the negative cyclic permutations are  $(4, 3, 2, 1)$ ,  $(3, 2, 1, 4)$ ,  $(2, 1, 4, 3)$  and  $(1, 4, 3, 2)$ . As in the case of three elements, given one of the eight permutations, our aim is to determine its parity and this requires knowing at least two elements in the permutation or equivalently knowing the values of the function for two different variables.

For a four level quantum system (ququart), we can use the initial state  $|\psi_2\rangle = (|1\rangle + i|2\rangle - |3\rangle - i|4\rangle)/2$  where  $|k\rangle$ 's are states of the ququart with vector representations  $|1\rangle = (1, 0, 0, 0)^T$ ,  $|2\rangle = (0, 1, 0, 0)^T$ ,  $|3\rangle = (0, 0, 1, 0)^T$ , and  $|4\rangle = (0, 0, 0, 1)^T$ . In this case we can use the standard quantum Fourier transformation [2] which can be viewed as a unitary matrix

$$U_{FT} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}, \quad (7)$$

in  $|k\rangle$ -basis so that  $|\psi_2\rangle = U_{FT}|2\rangle$ . We observe that the positive cyclic permutations  $(1, 2, 3, 4)$ ,  $(2, 3, 4, 1)$ ,  $(3, 4, 1, 2)$  and  $(4, 1, 2, 3)$  map  $|\psi_2\rangle$  onto  $|\psi_2\rangle$ ,  $-i|\psi_2\rangle$ ,  $-|\psi_2\rangle$ , and  $i|\psi_2\rangle$ , respectively. On the other hand, the negative cyclic permutations result in  $-i|\psi_4\rangle$ ,  $-|\psi_4\rangle$ ,  $i|\psi_4\rangle$ , and  $|\psi_4\rangle$ , respectively, where  $|\psi_4\rangle = U_{FT}|4\rangle$ . Therefore, by applying the inverse Fourier transformation  $U_{FT}^\dagger$  and checking the final state of the ququart, we can determine the parity of the cyclic permutation. Final state  $|2\rangle$  indicates that the permutation is even while  $|4\rangle$  means that the permutation is odd. Therefore, as in the case of qutrit, the quantum algorithm allows us to determine the parity of a cyclic permutation with single evaluation of the permutation function instead of two.

For four elements, we can formulate two more examples using other circular permutations. We can handle these new cases by redefining the Fourier transformation. For example, the

positive cyclic permutations  $(1, 3, 2, 4)$ ,  $(3, 2, 4, 1)$ ,  $(2, 4, 1, 3)$ ,  $(4, 1, 3, 2)$ , and the negative ones  $(4, 2, 3, 1)$ ,  $(2, 3, 1, 4)$ ,  $(3, 1, 4, 2)$  can be distinguished with single evaluation if we start with the state  $|\psi_2\rangle = (|1\rangle - |2\rangle + i|3\rangle - i|4\rangle)/2$ . The last eight members of the total  $4! = 24$  permutations, can be used to set up a similar problem.

From the above generalizations, we can deduce that the essence of the algorithm is to design a circuit so that output states are grouped according to the computational task where final states are described by the same vectors up to a phase factor. For this type of generalizations, it is clear that speed-up factor will be two as in the case of a single qutrit.

## V. CONCLUSION

We have shown that a single qutrit system can be used to implement a quantum algorithm which provides a two to one speed-up in determining parity of a permutation. Even though the model problem is not one of the most important computational tasks and the speed-up is not exponential when it is generalized to higher dimensional cases, the algorithm is important since it provides an example for quantum computation without entanglement. In fact, it is probably the simplest possible quantum algorithm.

A qutrit is known to be the smallest system that can be used to demonstrate contextual nature of quantum mechanics. Presence of a quantum algorithm for a single qutrit system points the importance of contextuality in quantum computing. Our result is in agreement with the recent observation that contextuality is the basic resource to perform certain computations.

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